

ON THE EXISTENCE OF A KAZANTZIS–KRAVARIS / LUENBERGER OBSERVER

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Abstract. We state sufficient conditions for the existence, on a given open set, of the extension, to non linear systems, of the Luenberger observer as it has been proposed by Kazantzis and Kravaris. We prove it is sufficient to choose the dimension of the system, giving the observer, less than or equal to $2 +$ twice the dimension of the state to be observed. We show that it is sufficient to know only an approximation of the solution of a PDE, needed for the implementation. We establish a link with high gain observers. Finally we extend our results to systems satisfying an unboundedness observability property.

Key words. Nonlinear osbservers, Luenberger observers, High gain observers

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1. Introduction. We consider the system :

$$(1.1) \quad \dot{x} = f(x) \quad , \quad y = h(x)$$

with state x in \mathbb{R}^n and output y in \mathbb{R}^p and where the functions f and h are sufficiently smooth. We are concerned with the problem of existence of an observer for x from the measurement y .

In a seminal paper [13], Kazantzis and Kravaris have proposed to extend to the nonlinear case the primary observer introduced by Luenberger in [17] for linear systems. Following this suggestion, the estimate \hat{x} of x is obtained as the output of the dynamical system :

$$(1.2) \quad \dot{z} = Az + B(y) \quad , \quad \hat{x} = T^*(z) \quad ,$$

with state z (a complex matrix) in $\mathbb{C}^{m \times p}$ and where A is a Hurwitz complex matrix and B and T^* are sufficiently smooth functions.

In the following we state sufficient conditions on f and h such that we can find (A, B, m) for which there exists T^* guaranteeing the convergence of \hat{x} to x .

To ease readability, we have divided the paper into two parts. In a first part, we introduce and state our main results which are proved in the second part. Our first result gives a sufficient condition on f , h , A and B implying the existence of T^* providing an appropriate observer. This condition involves a partial differential equation whose solution should be injective. In our second result, we propose a set of assumptions guaranteeing the existence of a solution for this equation. Our third and fourth results give two sufficient conditions implying the injectivity property of this solution. Our fifth result shows that an observer can already be obtained if we know only an appropriate approximation of this solution. This latter result allows us to propose a new insight in the standard high gain observer. Finally we claim that all these statements can be extended to the case where the system satisfies an unboundedness observability property.

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Some notations : We assume the functions f and h in (1.1) are at least locally Lipschitz. So, for each x in \mathbb{R}^n , there exists a unique solution $X(x, t)$ to (1.1), with x as initial condition.

Given an open set \mathcal{O} of \mathbb{R}^n , for each x in \mathcal{O} , we denote by $(\sigma_{\mathcal{O}}^-(x), \sigma_{\mathcal{O}}^+(x))$ the maximal interval of definition of the solution $X(x, t)$ conditioned to take values in \mathcal{O} .

For a set S , we denote by $\text{cl}(S)$ its closure and by $S + \delta$ the open set :

$$S + \delta = \{x \in \mathbb{R}^n : \exists x' \in S : |x - x'| < \delta\} = \bigcup_{x \in S} \mathcal{B}_{\delta}(x) ,$$

where $\mathcal{B}_{\delta}(x)$ denotes the open ball with center x and radius δ .

By $L_f V$ we denote the Lie derivative of V when it makes sense, i.e. :

$$L_f V(x) = \lim_{h \rightarrow 0} \frac{V(X(x, h)) - V(x)}{h} .$$

Finally, B_{1m} denotes the following vector in \mathbb{R}^m

$$(1.3) \quad B_{1m} = (1 \quad \dots \quad 1)^T$$

2. Results and comments.

2.1. Existence of a Kazantzis-Kravaris / Luenberger observer. In [13], m , the row dimension of z , is chosen equal to n , the dimension of x , and T^* is the inverse T^{-1} of a function T , solution of the following partial differential equation :

$$(2.1) \quad \frac{\partial T}{\partial x}(x) f(x) = A T(x) + B(h(x)) .$$

The rationale for this equation, as more emphasized in [15] (see also [18]), is that, if T is a diffeomorphism satisfying (2.1), then the change of coordinates :

$$(2.2) \quad \zeta = T(x)$$

allows us to rewrite the dynamics (1.1) equivalently as :

$$\dot{\zeta} = A \zeta + B(h(T^{-1}(\zeta))) \quad , \quad y = h(T^{-1}(\zeta)) .$$

We then have :

$$\overbrace{z - \zeta}^{\dot{}} = A(z - \zeta) .$$

A being Hurwitz, z in (1.2) is the state of an asymptotically convergent observer of $\zeta = T(x)$. Then, if the function $T^* = T^{-1}$ is uniformly continuous, $\hat{x} = T^*(z)$ is an asymptotically convergent observer of $x = T^*(\zeta) = T^*(T(x))$.

This way of finding the function T^* has motivated active research on the problem of the existence of an analytic and invertible solution to (2.1) (see [13, 15] for instance). But, it turns out that having a (weak) solution to (2.1) which is only continuous and uniformly injective is already sufficient. By sufficiency we mean, here, that an observer is appropriate if we have convergence to zero of the observation error associated to any solution which remains in a given open set \mathcal{O} . For the latter, we need :

DEFINITION 2.1 (Completeness within \mathcal{O}). *The system (1.1) is forward (resp. backward) complete within \mathcal{O} if we have the implication, for each x in \mathcal{O} ,*

$$(2.3) \quad \sigma_{\mathcal{O}}^+(x) < +\infty \quad \implies \quad \sigma_{\mathcal{O}}^+(x) < \sigma_{\mathbb{R}^n}^+(x) .$$

In other words, completeness within \mathcal{O} says that any solution $X(x, t)$ which exits \mathcal{O} in finite time must cross the boundary of \mathcal{O} (at a finite distance). An usual case where this property holds is when f has an at most linear growth within \mathcal{O} .

THEOREM 2.2 (Sufficient condition of existence of an observer). *Assume, the system (1.1) is forward complete within \mathcal{O} and there exist an integer m , a Hurwitz complex $m \times m$ matrix A and functions $T : \mathbf{cl}(\mathcal{O}) \rightarrow \mathbb{C}^{m \times p}$, continuous, $B : \mathbb{R}^p \rightarrow \mathbb{C}^{m \times p}$, continuous, and ρ , of class \mathcal{K}_∞ , satisfying :*

$$(2.4) \quad L_f T(x) = AT(x) + B(h(x)) \quad \forall x \in \mathcal{O} ,$$

$$(2.5) \quad |x_1 - x_2| \leq \rho(|T(x_1) - T(x_2)|) \quad \forall (x_1, x_2) \in \mathbf{cl}(\mathcal{O})^2 .$$

Under these conditions, there exists a continuous function $T^ : \mathbb{C}^{m \times p} \rightarrow \mathbf{cl}(\mathcal{O})$ such that, for each x in \mathcal{O} and z in $\mathbb{C}^{m \times p}$ the (unique) solution $(X(x, t), Z(x, z, t))$ of :*

$$(2.6) \quad \dot{x} = f(x) \quad , \quad \dot{z} = Az + B(h(x))$$

is right maximally defined on $[0, \sigma_{\mathbb{R}^n}^+(x))$. Moreover, we have the implication :

$$(2.7) \quad \sigma_{\mathcal{O}}^+(x) = \sigma_{\mathbb{R}^n}^+(x) \quad \implies \quad \lim_{t \rightarrow \sigma_{\mathbb{R}^n}^+(x)} |T^*(Z(x, z, t)) - X(x, t)| = 0 .$$

Remark 2.3 :

1. With the forward completeness within \mathcal{O} (2.3), the condition on the left in (2.7) implies that the solution $X(x, t)$ never exits \mathcal{O} and so :

$$(2.8) \quad \sigma_{\mathcal{O}}^+(x) = \sigma_{\mathbb{R}^n}^+(x) = +\infty .$$

2. Theorem 2.2 extends readily to the case where a) y is a scalar, b) the state x can be decomposed in $x = (\xi_1, \xi_2)$ and satisfies :

$$\dot{\xi}_1 = f_1(\xi_1, u) + h(\xi_2) \quad , \quad \dot{\xi}_2 = f_2(\xi_2) \quad , \quad y = \xi_1$$

and c) the function B can be chosen linear. In this case the observer is implemented as the reduced order observer :

$$\overline{\dot{z} - By} = Az + Bf_1(y, u) \quad , \quad \hat{\xi}_2 = T^*(z) .$$

Assuming we have a continuous function T satisfying (2.4), to implement the observer, we have to find a function T^* satisfying :

$$|T^*(z) - x| \leq \rho^*(|z - T(x)|) \quad \forall (x, z) \in \mathcal{O} \times \mathbb{C}^{m \times p}$$

for some function ρ^* of class \mathcal{K}_∞ . As shown by Kreisselmeier and Engel in [14], such a function T^* exists if T is continuous and uniformly injective as prescribed by (2.5).

In conclusion, a Kazantzis-Kravaris / Luenberger observer exists mainly if we can find a continuous function T solving (2.4) and uniformly injective in the sense of (2.5).

2.2. Existence of T solving (2.4). To exhibit conditions guaranteeing the existence of a function T solution of (2.4), we abandon the interpretation above of a change of coordinates (see (2.2)) and come back to the original idea in [17] (see also [13] and [4]) of dynamic extension. Namely, we consider the augmented system (2.6). Because of its triangular structure and the fact that A is Hurwitz, we may expect

this system to have, at least maybe only locally, an exponentially attractive invariant manifold in the augmented (x, z) space which could even be described as the graph of a function as :

$$\{(x, z) \in \mathbb{R}^n \times \mathbb{C}^{m \times p} : z = T(x)\}.$$

In this case, the function T would satisfy the following identity, for all t in the domain of definition of the solution $(X(x, t), Z((x, z), t))$ of (2.6) issued from (x, z) (compare with [18, Definition 5]),

$$T(X(x, t)) = Z((x, T(x)), t),$$

or equivalently :

$$(2.9) \quad T(X(x, t)) = \exp(At)T(x) + \int_0^t \exp(As)B(h(X(x, s)))ds.$$

From this identity, (2.4) is obtained by derivation with respect to t . But, since we need (2.4) to hold only on \mathcal{O} , from (2.9), it is sufficient that T satisfies :

$$T(x) = \exp(-At)T(\check{X}(x, t)) - \int_0^t \exp(-As)B(h(\check{X}(x, s)))ds,$$

where $\check{X}(x, s)$ is a solution of the modified system :

$$(2.10) \quad \dot{x} = \check{f}(x) = \chi(x)f(x)$$

where $\chi : \mathbb{R}^n \rightarrow \mathbb{R}$ is an arbitrary locally Lipschitz function satisfying :

$$(2.11) \quad \chi(x) = 1 \quad \text{if } x \in \mathcal{O}, \quad \chi(x) = 0 \quad \text{if } x \notin \mathcal{O} + \delta_u,$$

for some positive real number δ_u . So, as standard in the literature on invariant manifolds, by letting t go to $-\infty$, we get the following candidate expression for T :

$$(2.12) \quad T(x) = \int_{-\infty}^0 \exp(-As)B(h(\check{X}(x, s)))ds.$$

The above non rigorous reasoning can be made correct as follows :

THEOREM 2.4 (Existence of T). *Assume the existence of a strictly positive real number δ_u such that the system (1.1) is backward complete within $\mathcal{O} + \delta_u$. Then, for each Hurwitz complex $m \times m$ matrix A , we can find a C^1 function $B : \mathbb{R}^p \rightarrow \mathbb{C}^{m \times p}$ such that the function $T : \text{cl}(\mathcal{O}) \rightarrow \mathbb{C}^{m \times p}$, given by (2.12), is continuous and satisfies (2.4).*

Remark 2.5 : All what is needed here about the function B is that it guarantees that the function $t \mapsto |\exp(-At)B(h(\check{X}(x, t)))|$ is exponentially decaying with t going to $-\infty$. So in particular (see Remark 3.1) when $\text{cl}(\mathcal{O})$ is bounded, B can be chosen simply as a linear function.

Approaching the problem from another perspective, Kreisselmeier and Engel have introduced in [14] this same expression (2.12) (but with X instead of \check{X} and B the identity function). Another link between [13] and [14] has been established in [16].

2.3. T injective. Assuming now we have at our disposal the continuous function T , we need to make sure that it is injective, if not uniformly injective as specified by (2.5). Here is where observability enters the game. Following [17], in [13, 15], when $m = n$, observability of the first order approximation at an equilibrium together with an appropriate choice of A and B is shown to imply injectivity of the solution T of (2.1) in a neighborhood of this equilibrium. In [14], uniform injectivity of T is obtained under the following two assumptions :

1. The past output path $t \mapsto h(X(x, t))$ is uniformly injective in x with the set of past output paths equipped with an exponentially weighted L^2 -norm.

2. The system (1.1) has finite complexity, i.e. there exists a (finite) number M of piecewise continuous function ϕ_i in $L^2(\mathbb{R}_-; \mathbb{R}^p)$ and a strictly positive real number δ such that we have, for each pair (x_1, x_2) in \mathcal{O}^2 ,

$$\begin{aligned} \sum_{i=1}^M \left[\int_{-\infty}^0 \exp(-\ell s) \phi_i(s)^T [h(X(x_1, s)) - h(X(x_2, s))] ds \right]^2 \\ \geq \delta \int_{-\infty}^0 \exp(-2\ell s) |h(X(x_1, s)) - h(X(x_2, s))|^2 ds . \end{aligned}$$

Our next result states, that, with the only assumption that the past output path $t \mapsto h(X(x, t))$ is injective in x , it is sufficient to choose $m = n + 1$ generic complex eigen values for A to get T injective. The specific injectivity condition we need is :

DEFINITION 2.6 (Backward \mathcal{O} -distinguishability). *There exists two strictly positive real number $\delta_\Upsilon < \delta_d$ such that, for each pair of distinct points x_1 and x_2 in $\mathcal{O} + \delta_\Upsilon$, there exists a time t , in $(\max\{\sigma_{\mathcal{O}+\delta_d}^-(x_1), \sigma_{\mathcal{O}+\delta_d}^-(x_2)\}, 0]$, such that we have :*

$$h(X(x_1, t)) \neq h(X(x_2, t)) .$$

This distinguishability assumption says that the present state x can be distinguished from other states in $\mathcal{O} + \delta_\Upsilon$ by looking at the past output path restricted to the negative time interval where the solution $X(x, t)$ is in $\mathcal{O} + \delta_d$.

THEOREM 2.7 (Injectivity). *Assume the system (1.1) is backward complete within $\mathcal{O} + \delta_u$ and backward \mathcal{O} -distinguishable with the corresponding δ_d in $(0, \delta_u)$. Assume also the existence of an injective C^1 function $b : \mathbb{R}^p \rightarrow \mathbb{C}^p$, a continuous function $M : \mathcal{O} + \delta_\Upsilon \rightarrow \mathbb{R}^+$, and a negative real number ℓ such that, for each x in $\mathcal{O} + \delta_\Upsilon$, the two functions $t \mapsto \exp(-\ell t)b(h(\check{X}(x, t)))$ and $t \mapsto \exp(-\ell t)\frac{\partial b \circ h \circ \check{X}}{\partial x}(x, t)$ satisfy, for each t in $(\check{\sigma}_{\mathbb{R}^n}^-(x), 0]$,*

$$(2.13) \quad |\exp(-\ell t)b(h(\check{X}(x, t)))| + |\exp(-\ell t)\frac{\partial b \circ h \circ \check{X}}{\partial x}(x, t)| \leq M(x) ,$$

where again \check{X} is a solution of (2.10), but this time with the function χ satisfying :

$$(2.14) \quad \chi(x) = 1 \quad \text{if } x \in \mathcal{O} + \delta_d \quad , \quad \chi(x) = 0 \quad \text{if } x \notin \mathcal{O} + \delta_u .$$

Under these conditions, there exists a subset S of \mathbb{C}^{n+1} of zero Lebesgue measure such that the function $T : \mathbf{cl}(\mathcal{O}) \rightarrow \mathbb{C}^{(n+1) \times p}$ defined, with the notation (1.3), by :

$$(2.15) \quad T(x) = \int_{-\infty}^0 \exp(-As) B_{1m} b(h(\check{X}(x, s))) ds ,$$

is injective provided A is a diagonal matrix with $n + 1$ complex eigen values λ_i arbitrarily chosen in $\mathbb{C}^{n+1} \setminus S$ and with real part strictly smaller than ℓ .

Remark 2.8 :

1. Condition (2.13) holds for instance if f , h and b have bounded derivative on $\text{cl}(\mathcal{O} + \delta_Y)$ (see [16]).
2. Theorem 2.7 gives injectivity, not uniform injectivity. As already mentioned, if \mathcal{O} is bounded, continuity and the former imply the latter.

Following Theorem 2.7, for any generic choice of $n+1$ complex eigenvalues for the matrix A , the function T given by (2.15) (or equivalently (2.12)) is injective. This says that the (real) row dimension of z is $m = 2n + 2$. It is a well known fact in observer theory that it is generically sufficient to extract $m = 2n + 1$ pieces of information from the output path (with h generically chosen) to observe a state of dimension n (see for instance [1, 23, 10, 7, 22]). It can be understood from the adage that, the relation $T(x_1) = T(x_2)$ between the two states x_1 and x_2 in \mathbb{R}^n , i.e. for $2n$ unknowns, has generically the unique trivial solution $x_1 = x_2$ if we have strictly more than $2n$ equations, i.e. $T(x)$ has strictly more than $2n$ components.

2.4. Injectivity in the case of complete observability. Another setup where injectivity can be obtained is when we have complete observability. Namely we can find a row dimension m and a function $b : y \in \mathbb{R}^p \mapsto b(y) = (b_1(y), \dots, b_p(y)) \in \mathbb{R}^p$ so that the following function $H : \mathbb{R}^n \rightarrow \mathbb{R}^{m \times p}$ is injective when restricted to $\text{cl}(\mathcal{O})$:

$$(2.16) \quad H(x) = \begin{pmatrix} b_1(h(x)) & \dots & b_p(h(x)) \\ L_f b_1(h(x)) & \dots & L_f b_p(h(x)) \\ \vdots & \vdots & \vdots \\ L_f^{m-1} b_1(h(x)) & \dots & L_f^{m-1} b_p(h(x)) \end{pmatrix}.$$

Here $L_f^i h$ denotes the i th iterate Lie derivative, i.e. $L_f^{i+1} h = L_f(L_f^i h)$. Of course, for this to make sense, the functions b , f and h must be sufficiently smooth. This setup has been popularized and studied in deep details by Gauthier and his coworkers (see [11] and the references therein, see also [18]). In particular, it is established in [10] that, when $p = 1$, for any generic pair (f, h) in (1.1), it is sufficient to pick $m = 2n + 1$.

With a Taylor expansion of the output path at $t = 0$, we see that the injectivity of H implies that the function which associates the initial condition x to the output path, restricted to a very small time interval, is injective. This property is nicely exploited by observers with very fast dynamics as high gain observers (see [11]). Specifically, we have :

THEOREM 2.9 (Injectivity in the case of complete observability). *Assume the existence of a sufficiently smooth function $b : \mathbb{R}^p \rightarrow \mathbb{C}^p$ such that, for the function H defined in (2.16), there exist a positive real number L and a class \mathcal{K}_∞ function ρ such that we have :*

$$(2.17) \quad |L_f^m b(h(x_1)) - L_f^m b(h(x_2))| \leq L |H(x_1) - H(x_2)| \quad \forall (x_1, x_2) \in \text{cl}(\mathcal{O})^2,$$

$$(2.18) \quad |x_1 - x_2| \leq \rho(|H(x_1) - H(x_2)|) \quad \forall (x_1, x_2) \in \text{cl}(\mathcal{O})^2.$$

Then, for any diagonal Hurwitz complex $m \times m$ matrix A , with m the row dimension of H , there exists a real number k^* such that, for any k strictly larger than k^* , there exists a function $T : \text{cl}(\mathcal{O}) \rightarrow \mathbb{C}^{m \times p}$ which is continuous, uniformly injective and satisfies (see (1.3)) :

$$(2.19) \quad L_f T(x) = kAT(x) + B_{1m}b(h(x)) \quad \forall x \in \mathcal{O}.$$

2.5. Approximation. Fortunately for the implementation of the observer, knowing a function T satisfying (2.4) only approximately is sufficient. But, in this case, we have to modify the observer dynamics.

THEOREM 2.10 (Approximation). *Assume the system (1.1) is forward complete within \mathcal{O} . Assume also the existence of an integer m , a Hurwitz complex $m \times m$ matrix A and functions $T_a : \mathbf{cl}(\mathcal{O}) \rightarrow \mathbb{C}^{m \times p}$, continuous, $B : \mathbb{R}^p \rightarrow \mathbb{C}^{m \times p}$, continuous, and ρ of class \mathcal{K}_∞ , such that :*

$$(2.20) \quad |x_1 - x_2| \leq \rho(|T_a(x_1) - T_a(x_2)|) \quad \forall (x_1, x_2) \in \mathbf{cl}(\mathcal{O})^2 ,$$

the function $L_f T_a$ is well defined on \mathcal{O} and the function $\mathfrak{E} : \mathbf{cl}(\mathcal{O}) \rightarrow \mathbb{C}^{m \times p}$ defined as :

$$(2.21) \quad \mathfrak{E}(x) = L_f T_a(x) - [AT_a(x) + B(h(x))] \quad \forall x \in \mathcal{O}$$

satisfies :

$$(2.22) \quad |\mathfrak{E}(x_1) - \mathfrak{E}(x_2)| \leq N |T_a(x_1) - T_a(x_2)| \quad \forall (x_1, x_2) \in \mathbf{cl}(\mathcal{O})^2 ,$$

where N is a positive real number satisfying :

$$(2.23) \quad 2N \lambda_{\max}(P) < 1 ,$$

with $\lambda_{\max}(P)$ the largest eigenvalue of the Hermitian matrix P solution of :

$$(2.24) \quad \overline{A}^\top P + PA = -I .$$

Under these conditions, there exists a function $T_a^* : \mathbb{C}^{m \times p} \rightarrow \mathbf{cl}(\mathcal{O})$ and a locally Lipschitz function $\mathfrak{F} : \mathbb{C}^{m \times p} \rightarrow \mathbb{C}^{m \times p}$ such that, for each x in \mathcal{O} and z in $\mathbb{C}^{m \times p}$ each solution $(X(x, t), Z(x, z, t))$ of :

$$(2.25) \quad \dot{x} = f(x) \quad , \quad \dot{z} = Az + \mathfrak{F}(z) + B(h(x))$$

is right maximally defined on $[0, \sigma_{\mathbb{R}^n}^+(x))$. Moreover, we have the implication :

$$(2.26) \quad \sigma_{\mathcal{O}}^+(x) = \sigma_{\mathbb{R}^n}^+(x) \quad \implies \quad \lim_{t \rightarrow \sigma_{\mathbb{R}^n}^+(x)} |T_a^*(Z(x, z, t)) - X(x, t)| = 0 .$$

Remark 2.11 :

1. In (2.21), \mathfrak{E} represents the error in (2.4) given by the approximation T_a of T . This error should not be too large in an incremental sense as specified by (2.22) and (2.23). This indicates that one way to approximate T is to look for T_a in a set of functions minimizing the L^∞ norm on $\mathbf{cl}(\mathcal{O})$ of the gradient of the associated error \mathfrak{E} . In particular, in the case where \mathcal{O} is bounded, it follows from Weierstrass Approximation Theorem that we can always choose a Hurwitz complex matrix A and a linear function B so that the constraint (2.23) can be satisfied by restricting ourself to choose the function T_a as a polynomial in x .

2. The function \mathfrak{F} in the observer (2.25) can be chosen as any Lipschitz extension of $\mathfrak{E} \circ T_a^*$ outside $T_a(\mathbf{cl}(\mathcal{O}))$. This is very similar to what is done in [18] where a constructive procedure for this extension is proposed. Fortunately, this Lipschitz extension is not needed in the case where the function \mathfrak{E} satisfies :

$$|\mathfrak{E}(x_1) - \mathfrak{E}(x_2)| \leq \frac{N}{4} \rho^{-1}(|x_1 - x_2|) \quad \forall (x_1, x_2) \in \mathbf{cl}(\mathcal{O})^2 ,$$

where ρ is the function satisfying (2.20). In this case we take simply (see (3.33)) :

$$\mathfrak{F}(z) = \mathfrak{E}(T_a^*(z)) \quad \forall z \in \mathbb{C}^{m \times p} .$$

The combination of Theorems 2.9 and 2.10 gives us a new insight in the classical high gain observer of order m as studied in [10] or [18] for instance. Specifically, we have :

COROLLARY 2.12 (High gain Observer). *Assume the system (1.1) is forward complete within \mathcal{O} and there exist a sufficiently smooth function $b : \mathbb{R}^p \rightarrow \mathbb{C}^p$, a class \mathcal{K}_∞ function ρ and a positive real number L such that (2.17) and (2.18) hold with H and m given by (2.16). Under these conditions, for any diagonal Hurwitz complex $m \times m$ matrix A , there exists a real number k^* such that, for any real number k strictly larger than k^* , there exist a function $T_a^* : \mathbb{C}^{m \times p} \rightarrow \mathbf{cl}(\mathcal{O})$, left inverse on $T_a(\mathbf{cl}(\mathcal{O}))$ of the function $T_a : \mathbf{cl}(\mathcal{O}) \rightarrow \mathbb{C}^{m \times p}$ defined as :*

$$(2.27) \quad T_a(x) = - \sum_{i=1}^m (kA)^{-i} B_{1m} L_f^{i-1} b(h(x)) ,$$

and a function $\mathfrak{F} : \mathbb{R}^{m \times p} \rightarrow \mathbb{R}^{m \times p}$ such that, for each x in \mathcal{O} and z in $\mathbb{R}^{m \times p}$ each solution $(X(x, t), Z(x, z, t))$ of :

$$(2.28) \quad \dot{x} = f(x) \quad , \quad \dot{z} = kAz + \mathfrak{F}(z) + B_{1m} b(h(x))$$

is right maximally defined on $[0, \sigma_{\mathbb{R}^n}^+(x))$. Moreover, we have the implication :

$$(2.29) \quad \sigma_{\mathcal{O}}^+(x) = \sigma_{\mathbb{R}^n}^+(x) \quad \implies \quad \lim_{t \rightarrow \sigma_{\mathbb{R}^n}^+(x)} |T_a^*(Z(x, z, t)) - X(x, t)| = 0 .$$

Remark 2.13 : When \mathcal{O} is bounded and H is injective, uniform injectivity (2.18) and forward completeness within \mathcal{O} hold necessarily. Thus, in this case, we recover [18, Lemma 1].

2.6. Extension to boundedness observability. Completeness is a severe restriction. Instead, it is proved in [4] that a necessary condition for the existence of an observer providing the convergence to zero of the observation error within the domain of definition of the solutions is the forward unboundedness observability property.

DEFINITION 2.14 (Unboundedness observable within \mathcal{O}). *The system (1.1) is forward (resp. backward) unboundedness observable within \mathcal{O} if there exists a proper and C^1 function $V_f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ (resp. $V_b : \mathbb{R}^n \rightarrow \mathbb{R}_+$) and a continuous function $\gamma_f : \mathbb{R}^p \rightarrow \mathbb{R}_+$ (resp. $\gamma_b : \mathbb{R}^p \rightarrow \mathbb{R}_+$) such that :*

$$(2.30) \quad \begin{aligned} L_f V_f(x) &\leq V_f(x) + \gamma_f(h(x)) & \forall x \in \mathcal{O} , \\ (\text{resp. } L_f V_b(x) &\geq -V_b(x) - \gamma_b(h(x)) & \forall x \in \mathcal{O} .) \end{aligned}$$

Fortunately, all our previous results still hold if completeness is replaced by unboundedness observability but provided¹ :

1. the observer is modified in :

$$\dot{z} = \gamma(y) [Az + B(y)] \quad , \quad \hat{x} = T^*(z) ,$$

where γ is a C^1 function satisfying :

$$\gamma(h(x)) \geq 1 + \gamma_f(h(x)) \quad (\text{resp. and } \gamma(h(x)) \geq 1 + \gamma_b(h(x))) \quad \forall x \in \mathbf{cl}(\mathcal{O}) .$$

As suggested in [4], the introduction of γ takes care of possible finite escape time. This has nothing in common with the objective of error dynamics linearization as considered in [19].

¹The interested reader will find in [3] the precise statements of the corresponding results.

2. In most occurrences, e.g. (2.4), (2.10), (2.16), (2.19), (2.21), \dots , f is replaced by f_γ defined as :

$$f_\gamma(x) = \frac{f(x)}{\gamma(h(x))} .$$

3. Proofs.

3.1. Proof of Theorem 2.2. Because of the triangular structure of the system (2.6), for each x in \mathcal{O} and z in $\mathbb{C}^{m \times p}$, the component $Z(x, z, t)$ of the corresponding solution of this system is defined as long as $h(X(x, t))$ is defined. So this solution $(X(x, t), Z(x, z, t))$ is right maximally defined on the same interval $[0, \sigma_{\mathbb{R}^n}^+(x))$ as $X(x, t)$, solution of (1.1).

Let us now restrict our attention to points x in \mathcal{O} satisfying the condition on the left in (2.7). In this case, with the forward completeness within \mathcal{O} , we have (2.8). On the other hand, from (2.4) and (2.6), we obtain, for each x in \mathcal{O} , z in $\mathbb{C}^{m \times p}$ and t in $[0, \sigma_{\mathbb{R}^n}^+(x))$,

$$(3.1) \quad T(X(x, t)) - Z(x, z, t) = \exp(At) (T(x) - z) .$$

As A is a Hurwitz matrix, this and (2.8) yield :

$$\lim_{t \rightarrow \sigma_{\mathbb{R}^n}^+(x)} |Z(x, z, t) - T(X(x, t))| = 0 .$$

From this, the implication (2.7) follows readily if there exist a continuous function $T^* : \mathbb{C}^{m \times p} \rightarrow \mathbf{cl}(\mathcal{O})$ and a class \mathcal{K}_∞ function $\rho^* : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying :

$$(3.2) \quad |T^*(z) - x| \leq \rho^*(z - T(x)) \quad \forall z \in \mathbb{C}^{m \times p}, \forall x \in \mathbf{cl}(\mathcal{O}) ,$$

To find such functions, we first remark, as in [14], that (2.5) and completeness of $\mathbb{C}^{m \times p}$ and \mathbb{R}^n imply that $T(\mathbf{cl}(\mathcal{O}))$ is a closed subset of $\mathbb{C}^{m \times p}$. It follows that, for each z in $\mathbb{C}^{m \times p}$, the infimum, in x in $\mathbf{cl}(\mathcal{O})$, of $|T(x) - z|$ is achieved by at least one point, denoted $T_p^*(z)$ (in $\mathbf{cl}(\mathcal{O})$). This defines a function $T_p^* : \mathbb{C}^{m \times p} \rightarrow \mathbf{cl}(\mathcal{O})$ satisfying :

$$(3.3) \quad T(T_p^*(z)) = z \quad \forall z \in T(\mathbf{cl}(\mathcal{O})) ,$$

$$(3.4) \quad |T(T_p^*(z)) - z| \leq |T(x) - z| \quad \forall z \in \mathbb{C}^{m \times p}, \forall x \in \mathbf{cl}(\mathcal{O}) .$$

With (2.5), (3.3) implies that the restriction T_p^* to $T(\mathbf{cl}(\mathcal{O}))$ is continuous. Also, with the triangle inequality, (3.4) gives, for each z in $\mathbb{C}^{m \times p}$ and each x in $\mathbf{cl}(\mathcal{O})$,

$$(3.5) \quad |x - T_p^*(z)| \leq \rho(|T(x) - z| + |z - T(T_p^*(z))|) \leq \rho(2|T(x) - z|) .$$

Now we build the function T^* by smoothing out T_p^* . For each z in $\mathbb{C}^{m \times p}$, let

$$\epsilon(z) = \frac{1}{2} \inf_{x \in \mathbf{cl}(\mathcal{O})} |T(x) - z| .$$

$(\mathcal{B}_{\epsilon(z)}(z))_{z \in \mathbb{C}^{m \times p} \setminus T(\mathbf{cl}(\mathcal{O}))}$ is a covering of the open set $\mathbb{C}^{m \times p} \setminus T(\mathbf{cl}(\mathcal{O}))$ by open subsets. From Lindelöf Theorem (see [6, Lemma 4.1] for instance), there exists a sequence $\{z_i\}_{i \in \mathbb{N}}$ such that $\{\mathcal{B}_{\epsilon(z_i)}(z_i)\}_{i \in \mathbb{N}}$ is a countable and locally finite covering

by open subsets of $\mathbb{C}^{m \times p} \setminus T(\text{cl}(\mathcal{O}))$. For each x in $\text{cl}(\mathcal{O})$, each z_i in $\{z_i\}_{i \in \mathbb{N}}$ and each z in $\mathcal{B}_{\epsilon(z_i)}(z_i)$, we have :

$$|z_i - z| < \epsilon(z_i) \leq \frac{1}{2} |T(x) - z_i| \leq \frac{1}{2} [|T(x) - z| + |z - z_i|] \leq |T(x) - z| .$$

With (3.5), this gives :

$$|x - T_p^*(z_i)| \leq \rho(2|T(x) - z_i|) \leq \rho(2(|T(x) - z| + |z - z_i|)) \leq \rho(4|T(x) - z|) .$$

From [6, Theorem IV.4.4], we know that there exists a countable set of C^∞ functions $\{\phi_i\}_{i \in \mathbb{N}} : \mathbb{C}^{m \times p} \setminus T(\text{cl}(\mathcal{O})) \rightarrow [0, 1]$ satisfying, for each z in $\mathbb{C}^{m \times p} \setminus T(\text{cl}(\mathcal{O}))$,

$$\sum_i \phi_i(z) = 1 \quad , \quad \phi_i(z) = 0 \quad \forall z \notin \mathcal{B}_{\epsilon(z_i)}(z_i) .$$

We define the function $T^* : \mathbb{C}^{m \times p} \rightarrow T(\text{cl}(\mathcal{O}))$ as :

$$\begin{aligned} T^*(z) &= \sum_i \phi_i(z) T_p^*(z_i) \quad \text{if } z \in \mathbb{C}^{m \times p} \setminus T(\text{cl}(\mathcal{O})) , \\ &= T_p^*(z) \quad \text{if } z \in T(\text{cl}(\mathcal{O})) . \end{aligned}$$

It is continuous when restricted to the open set $\mathbb{C}^{m \times p} \setminus T(\text{cl}(\mathcal{O}))$ and to the closed set $T(\text{cl}(\mathcal{O}))$. Also, for each z in $\mathbb{C}^{m \times p} \setminus T(\text{cl}(\mathcal{O}))$ and each x in $\text{cl}(\mathcal{O})$, we get :

$$\begin{aligned} |T^*(z) - x| &= \left| \sum_i \phi_i(z) T_p^*(z_i) - x \right| \leq \sum_i \phi_i(z) |T_p^*(z_i) - x| , \\ (3.6) \quad &\leq \sum_i \phi_i(z) \rho(4|z - T(x)|) \leq \rho(4|z - T(x)|) . \end{aligned}$$

And, for each z in $T(\text{cl}(\mathcal{O}))$ and each x in $\text{cl}(\mathcal{O})$, we get readily from (2.5) and (3.3) :

$$(3.7) \quad |T^*(z) - x| = |T_p^*(z) - x| \leq \rho(|T(T_p^*(z)) - T(x)|) = \rho(|z - T(x)|) .$$

With (3.6) and (3.7), (3.2) is established. This proves also that T^* is continuous on whole $\mathbb{C}^{m \times p}$.

3.1.1. Proof of Theorem 2.4. With [12, Corollary I.4.7], we know it exists a locally Lipschitz function $\chi : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying (2.11). It follows that the function \check{f} in (2.10) is locally Lipschitz. Thus, for each x in \mathbb{R}^n there exists a unique solution $\check{X}(x, t)$ of (2.10), with initial condition x , maximally defined on $(\check{\sigma}_{\mathbb{R}^n}^-(x), \check{\sigma}_{\mathbb{R}^n}^+(x))$. Moreover, backward completeness within $\mathcal{O} + \delta_u$ of (1.1) implies backward completeness of (2.10), i.e. $\check{\sigma}_{\mathbb{R}^n}^-(x) = -\infty$. Following [2], this implies the existence of a proper and C^1 function $V_b : \mathbb{R}^n \rightarrow \mathbb{R}_+$ and a continuous function $\gamma_b : \mathbb{R}^p \rightarrow \mathbb{R}_+$ satisfying :

$$(3.8) \quad L_{\check{f}} V_b(x) \geq -V_b(x) - 1 \quad \forall x \in \mathbb{R}^n .$$

Let α be a strictly positive real number so that $A + \alpha I$ is a Hurwitz matrix. We define the function $W_b : \mathbb{R}^n \rightarrow \mathbb{R}$ as :

$$W_b(x) = (V_b(x) + 1)^\alpha ,$$

With the help of Gronwall's Lemma, (3.8) yields :

$$(3.9) \quad W_b(\check{X}(x, t)) \leq W_b(x) \exp(-\alpha t) \quad \forall x \in \mathbb{R}^n , \forall t \in (-\infty, 0] .$$

Since W_b is a proper function and h is continuous, we can find a C^1 function $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ of class \mathcal{K}_∞ and a real number β_0 such that, for each component h_i of h , we have :

$$|h_i(x)| \leq \beta(W_b(x)) + \beta_0 \quad \forall x \in \mathbb{R}^n .$$

Let $\check{\beta} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be the function defined as :

$$\check{\beta}(w) = \sqrt{w} + \beta(w) + \beta_0 .$$

This function is strictly increasing, C^1 on $(0, +\infty)$ and its derivative $\check{\beta}'$ satisfies :

$$\lim_{x \rightarrow 0} \check{\beta}'(x) = +\infty .$$

It admits an inverse $\check{\beta}^{-1} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which satisfies :

$$(3.10) \quad \check{\beta}^{-1}(|h_i(x)|) \leq W_b(x) \quad \forall x \in \mathbb{R}^n .$$

Moreover the function $\eta \mapsto \frac{\eta \check{\beta}^{-1}(|\eta|)}{|\eta|}$ is C^1 on $\mathbb{R} \setminus \{0\}$ and can be extended by continuity on \mathbb{R} as a C^1 injective function. So, with p arbitrary vectors b_j in \mathbb{R}^m , we define the function $B : \mathbb{R}^p \rightarrow \mathbb{R}^{m \times p}$ as :

$$B(h) = \begin{pmatrix} \frac{h_1 \check{\beta}^{-1}(|h_1|)}{|h_1|} b_1 & \dots & \frac{h_p \check{\beta}^{-1}(|h_p|)}{|h_p|} b_p \end{pmatrix} .$$

Since $A + \alpha I$ is a Hurwitz matrix, (3.9), (3.10) and the backward completeness imply :

1. The existence of strictly positive real numbers c_0 , c_1 and ε such that we have :

$$(3.11) \quad |\exp(-As) B(h(\check{X}(x, s)))| \leq c_0 |\exp(-As)| W_b(\check{X}(x, s)) ,$$

$$(3.12) \quad \leq c_1 W_b(x) \exp(\varepsilon s) \quad \forall (s, x) \in \mathbb{R}_- \times \mathbb{R}^n .$$

2. For each fixed s in \mathbb{R}_- , the function $x \mapsto \exp(-As) B(h(\check{X}(x, s)))$ is continuous.

So Lebesgue dominated convergence Theorem (see [8, Théorème (3.149)] for instance) implies that the following expression defines properly a continuous function $T : \mathbb{R}^n \rightarrow \mathbb{C}^{m \times p}$:

$$(3.13) \quad T(x) = \int_{-\infty}^0 \exp(-As) B(h(\check{X}(x, s))) ds .$$

Then, for each x in \mathbb{R}^n and for each t in $(-\infty, \check{\sigma}_{\mathbb{R}^n}^+(x))$, we get :

$$\begin{aligned} T(\check{X}(x, t)) - T(x) &= \int_{-\infty}^0 \exp(-As) B(h(\check{X}(\check{X}(x, t), s))) ds - T(x) , \\ &= \exp(At) \int_{-\infty}^t \exp(-Au) B(h(\check{X}(x, u))) du - T(x) , \\ &= (\exp(At) - I) T(x) + \exp(At) \int_0^t \exp(-Au) B(h(\check{X}(x, u))) du . \end{aligned}$$

Thus, we obtain, for all x in \mathbb{R}^n ,

$$(3.14) \quad \chi(x) L_f T(x) = L_{\check{f}} T(x) = \lim_{t \rightarrow 0} \frac{T(\check{X}(x, t)) - T(x)}{t} = AT(x) + B(h(x)) .$$

With (2.11), this implies (2.4) is satisfied. \square

Remark 3.1 :

1. For the case where A is diagonalizable, with eigen value λ_i , and where the vectors b_j are chosen so that the p pairs (A, b_j) are controllable, our expression for T gives for its i th component in the diagonalizing coordinates :

$$(3.15) \quad T_i(x) = \int_{-\infty}^0 \exp(-\lambda_i s) B_i(h(\check{X}(x, s))) ds$$

with :

$$B_i(h) = \left(\frac{h_1 \check{\beta}^{-1}(|h_1|)}{|h_1|} b_{i1} \quad \dots \quad \frac{h_p \check{\beta}^{-1}(|h_p|)}{|h_p|} b_{ip} \right)$$

where each b_{ij} is non zero. Note that each of the m rows of the function B is an injective function from \mathbb{R}^p to \mathbb{R}^p .

2. If \mathcal{O} is bounded, the function $s \in \mathbb{R}_- \mapsto h(\check{X}(x, s)) \in \mathbb{R}^p$ is a bounded function, uniformly in x in $\mathbf{cl}(\mathcal{O})$. It follows that the inequality (3.12) holds by choosing the function $\check{\beta}^{-1}$ simply as the identity function. This says that, in this case, the function B is linear.

3.2. Proof of Theorem 2.7. We first remark that backward \mathcal{O} -distinguishability property of the original system (1.1) implies the same property for the modified system (2.10). Then we need the following Lemma.

LEMMA 3.2 (Coron). *Let Ω and Υ be open subsets of \mathbb{C} and \mathbb{R}^{2n} respectively. Let $g : \Upsilon \times \Omega \rightarrow \mathbb{C}^p$ be a function which is holomorphic in λ for each x in Υ and C^1 in x for each λ in Ω . If, for each pair (x, λ) in $\Upsilon \times \Omega$ for which $g(x, \lambda)$ is zero we can find, for at least one of the p components g_j of g , an integer k satisfying :*

$$(3.16) \quad \frac{\partial^i g_j}{\partial \lambda^i}(x, \lambda) = 0 \quad \forall i \in \{0, \dots, k-1\} \quad , \quad \frac{\partial^k g_j}{\partial \lambda^k}(x, \lambda) \neq 0$$

then the following set has zero Lebesgue measure in \mathbb{C}^{n+1} :

$$(3.17) \quad S = \bigcup_{x \in \Upsilon} \{(\lambda_1, \dots, \lambda_{n+1}) \in \Omega^{n+1} : g(x, \lambda_1) = \dots = g(x, \lambda_{n+1}) = 0\} .$$

This result has been established by Coron in [7, Lemma 3.2] in a stronger form except for the very minor point that, here, g is not C^∞ in both x and λ . To make sure that this difference has no bad consequence and for the sake of completeness, we give an ad hoc proof in appendix.

To complete the proof of Theorem 2.7 all we have to do is to generate an appropriate function g satisfying all the required assumptions of this Lemma 3.2.

Let Ω and Υ be the following open subsets of \mathbb{C} and \mathbb{R}^{2n} , respectively :

$$\Omega = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) < \ell\} \quad , \quad \Upsilon = \{x = (x_1, x_2) \in (\mathcal{O} + \delta_\Upsilon)^2 : x_1 \neq x_2\} .$$

By following the same arguments as in the proof of Theorem 2.4, the backward completeness allows us to conclude :

$$\check{\sigma}_{\mathbb{R}^n}^-(x) = -\infty \quad \forall x \in \mathcal{O} + \delta_\Upsilon .$$

Then, with (2.13), we get, for each (x, λ, t) in $(\mathcal{O} + \delta_{\Upsilon}) \times \Omega \times (-\infty, 0]$,

$$\begin{aligned} |\exp(-\lambda t) b(h(\check{X}(x, t)))| &\leq \exp([\ell - \operatorname{Re}(\lambda)]t) |\exp(-\ell t) b(h(\check{X}(x, t)))| , \\ &\leq \exp([\ell - \operatorname{Re}(\lambda)]t) M(x) . \end{aligned}$$

So Lebesgue dominated convergence Theorem implies that, for each fixed λ in Ω , the expression

$$T_{\lambda}(x) = \int_{-\infty}^0 \exp(-\lambda s) b(h(\check{X}(x, s))) ds$$

defines properly a continuous function $T_{\lambda} : \mathcal{O} + \delta_{\Upsilon} \rightarrow \mathbb{C}^p$. With similar arguments (see [8, Théorème (3.150)] for instance), with (2.13), we can establish that this function is actually C^1 .

Now, let $\mathcal{DT} : (\mathcal{O} + \delta_{\Upsilon})^2 \times \Omega \rightarrow \mathbb{C}^p$ be the function defined as :

$$\begin{aligned} (3.18) \quad \mathcal{DT}(x, \lambda) &= T_{\lambda}(x_1) - T_{\lambda}(x_2) , \\ &= \int_{-\infty}^0 \exp(-\lambda s) [b(h(\check{X}(x_1, s))) - b(h(\check{X}(x_2, s)))] ds , \end{aligned}$$

with $x = (x_1, x_2)$. It is C^1 in x in $(\mathcal{O} + \delta_{\Upsilon})^2$ for each λ in Ω . Also, as proved in [21, chap 19, p. 367] with the help of Morera and Fubini Theorems, it is holomorphic in λ in Ω for each x in $(\mathcal{O} + \delta_{\Upsilon})^2$. Moreover, since we have, for each a in $(-\infty, \ell)$,

$$\int_{-\infty}^0 \exp(-2as) |b(h(\check{X}(x_1, s))) - b(h(\check{X}(x_2, s)))|^2 ds \leq \frac{M(x_1)^2 + M(x_2)^2}{2(\ell - a)} < +\infty ,$$

we can apply Plancherel Theorem to obtain, for each a in $(-\infty, \ell)$ and each x in $(\mathcal{O} + \delta_{\Upsilon})^2$,

$$(3.19) \quad \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\mathcal{DT}(x, a + is)|^2 ds = \int_{-\infty}^0 \exp(-2as) |b(h(\check{X}(x_1, s))) - b(h(\check{X}(x_2, s)))|^2 ds .$$

Now, for x in Υ , with the distinguishability property, continuity with respect to time and injectivity of b imply the existence of an open time interval (t_0, t_1) such that :

$$|b(h(\check{X}(x_1, s))) - b(h(\check{X}(x_2, s)))| > 0 \quad \forall s \in (t_0, t_1) .$$

It follows with (3.19) that we have :

$$\int_{-\infty}^{+\infty} |\mathcal{DT}(x, a + is)|^2 ds > 0 .$$

This says that, for each x in Υ , the function $\lambda \mapsto \mathcal{DT}(x, \lambda)$ is not identically equal zero on Ω . Since it is holomorphic, this implies that, for each (x, λ) in $\Upsilon \times \Omega$, we can find, for at least one of the p components \mathcal{DT}_j of \mathcal{DT} , an integer k satisfying :

$$\frac{\partial^i \mathcal{DT}_j}{\partial \lambda^i}(x, \lambda) = 0 \quad \forall i \in \{0, \dots, k-1\} , \quad \frac{\partial^k \mathcal{DT}_j}{\partial \lambda^k}(x, \lambda) \neq 0 .$$

So we can invoke Lemma 3.2 with \mathcal{D} as function g . With (3.18), it allows us to conclude that the following set S has zero Lebesgue measure in \mathbb{C}^{n+1} :

$$S = \bigcup_{(x_1, x_2) \in \Upsilon} \{(\lambda_1, \dots, \lambda_{n+1}) \in \Omega^{n+1} : T_{\lambda_i}(x_1) = T_{\lambda_i}(x_2) \quad \forall i \in \{1, \dots, n+1\}\} .$$

3.3. Proof of Theorem 2.9. Our first step consists in proposing a function T solution of (2.19). The definition (2.16) of H and the inequality (2.17), give, for each pair (x_1, x_2) in $\text{cl}(\mathcal{O})^2$,

$$\begin{aligned} |L_f H(x_1) - L_f H(x_2)| &\leq |H(x_1) - H(x_2)| + |L_f^m b(h(x_1)) - L_f^m b(h(x_2))| , \\ &\leq (1 + L)|H(x_1) - H(x_2)| . \end{aligned}$$

Also (2.18) implies that, for each Y in $H(\text{cl}(\mathcal{O}))$, there exists a unique x in $\text{cl}(\mathcal{O})$ solution of $Y = H(x)$. Hence we can define a Lipschitz function $F : H(\text{cl}(\mathcal{O})) \rightarrow \mathbb{R}^{m \times p}$ satisfying :

$$(3.20) \quad F(H(x)) = L_f H(x) \quad \forall x \in \text{cl}(\mathcal{O}) .$$

Furthermore, as in the proof of Theorem 2.2, continuity and uniform injectivity of the function H on $\text{cl}(\mathcal{O})$ as given by (2.18) imply that $H(\text{cl}(\mathcal{O}))$ is closed. Then it follows from Kirszbraun's Lipschitz extension Theorem (see [9, Theorem 2.10.43] for instance) that F can be extended as a function $\check{F} : \mathbb{R}^{m \times p} \rightarrow \mathbb{R}^{m \times p}$ satisfying :

$$(3.21) \quad |\check{F}(Y_1) - \check{F}(Y_2)| \leq (1 + L)|Y_1 - Y_2| \quad \forall (Y_1, Y_2) \in \mathbb{R}^{m \times p} \times \mathbb{R}^{m \times p} ,$$

$$(3.22) \quad \check{F}(Y) = F(Y) \quad \forall Y \in H(\text{cl}(\mathcal{O})) .$$

Let $\mathfrak{Y}(Y, t)$ denote a solution of the following system on $\mathbb{R}^{m \times p}$:

$$\dot{Y} = \check{F}(Y) .$$

With (3.21), such a solution is unique for each Y in $\mathbb{R}^{m \times p}$, defined on $(-\infty, +\infty)$ and satisfies, for some fixed matrix Y_0 in $\mathbb{R}^{m \times p}$ and for each pair (Y_1, Y_2) in $\mathbb{R}^{m \times p} \times \mathbb{R}^{m \times p}$,

$$\begin{aligned} |\mathfrak{Y}(Y_1, t) - Y_0| &\leq |Y_1 - Y_0| + \int_t^0 |\check{F}(\mathfrak{Y}(Y_1, s)) - \check{F}(Y_0)| ds - |\check{F}(Y_0)| t , \\ &\leq |Y - Y_0| + (1 + L) \int_t^0 |\mathfrak{Y}(Y, s) - Y_0| ds - |\check{F}(Y_0)| t , \\ |\mathfrak{Y}(Y_1, t) - \mathfrak{Y}(Y_2, t)| &\leq (1 + L) \int_t^0 |\mathfrak{Y}(Y_1, s) - \mathfrak{Y}(Y_2, s)| ds . \end{aligned}$$

With Gronwall inequality, this gives, for all $t \leq 0$,

$$(3.23) \quad |\mathfrak{Y}(Y, t) - Y_0| \leq \exp(-(1 + L)t) \left[|Y - Y_0| + \frac{|\check{F}(Y_0)|}{1 + L} \right] - \frac{\check{F}(Y_0)}{1 + L} ,$$

$$(3.24) \quad |\mathfrak{Y}(Y_1, t) - \mathfrak{Y}(Y_2, t)| \leq \exp(-(1 + L)t) |Y_1 - Y_2| .$$

So, in particular, we have, for each $t \leq 0$ and Y in $\mathbb{R}^{m \times p}$,

$$(3.25) \quad |\check{F}(\mathfrak{Y}(Y, t))| \leq (1 + L) \left(\exp(-(1 + L)t) \left[|Y - Y_0| + \frac{|\check{F}(Y_0)|}{1 + L} \right] \right) .$$

Hence, given any diagonal Hurwitz $m \times m$ matrix A , with eigen value λ_i , for each real number $k \geq \frac{1+L}{-\max_i \{\text{Re}(\lambda_i)\}}$, we can properly define a continuous function $\mathfrak{R} : \mathbb{R}^{m \times p} \rightarrow \mathbb{C}^m$ as :

$$(3.26) \quad \mathfrak{R}(Y) = \int_{-\infty}^0 \exp(-skA) B_{1m} \check{F}(\mathfrak{Y}(Y, s))_m ds ,$$

with the notation (1.3), and where $\check{F}(Y)_m$ denotes the m th row of $\check{F}(Y)$. As for (3.14), we can prove that we have :

$$L_{\check{F}}\mathfrak{R}(Y) = kA\mathfrak{R}(Y) + B_{1m}\check{F}(Y)_m \quad \forall Y \in \mathbb{R}^{m \times m}.$$

But, with (2.16), (3.20) and (3.22), this yields :

$$(3.27) \quad L_F\mathfrak{R}(H(x)) = kA\mathfrak{R}(H(x)) + B_{1m}L_f^m b(h(x)) \quad \forall x \in \mathbf{cl}(\mathcal{O}).$$

Let now $T : \mathbf{cl}(\mathcal{O}) \rightarrow \mathbb{R}^n$ be the continuous function defined as :

$$(3.28) \quad T(x) = (kA)^{-m}\mathfrak{R}(H(x)) - K^{-1}SH(x),$$

with the notations :

$$(3.29) \quad S = \begin{pmatrix} \lambda_1^{-1} & \dots & \lambda_1^{-m} \\ \vdots & \vdots & \vdots \\ \lambda_m^{-1} & \dots & \lambda_m^{-m} \end{pmatrix}, \quad K = \text{diag}(k, \dots, k^m).$$

We want to show that T is a solution of (2.19). We have :

$$K^{-1}SH(x) = \begin{pmatrix} \sum_{i=1}^m (k\lambda_1)^{-i} L_f^{i-1} b(h(x)) \\ \vdots \\ \sum_{i=1}^m (k\lambda_m)^{-i} L_f^{i-1} b(h(x)) \end{pmatrix}.$$

Thus, for each x in \mathbb{R}^n , we get :

$$(3.30) \quad K^{-1}SL_f H(x) = kAK^{-1}SH(x) - B_{1m}b(h(x)) + (kA)^{-m}B_{1m}L_f^m b(h(x)).$$

In view of (3.28), it remains to compute the Lie derivatives of $(kA)^{-m}\mathfrak{R}(H(x))$. From (2.11), (2.16), (3.20) and (3.22), we get the identity :

$$\mathfrak{Y}(H(x), t) = H(\check{X}(x, t)) \quad \forall t \in (\check{\sigma}_{\mathcal{O}}^-(x), \check{\sigma}_{\mathcal{O}}^+(x)) \quad \forall x \in \mathcal{O}.$$

This gives readily, for all t in $(\check{\sigma}_{\mathcal{O}}^-(x), \check{\sigma}_{\mathcal{O}}^+(x))$ and x in \mathcal{O} ,

$$\mathfrak{R}(\mathfrak{Y}(H(x), t)) - \mathfrak{R}(H(x)) = \mathfrak{R}(H(\check{X}(x, t)) - \mathfrak{R}(H(x)))$$

and therefore :

$$L_F\mathfrak{R}(H(x)) = L_f\mathfrak{R}(H(x)) \quad \forall x \in \mathcal{O}.$$

By continuity this identity extends to $\mathbf{cl}(\mathcal{O})$. So, with (3.27), we get :

$$L_f\mathfrak{R}(H(x)) = kA\mathfrak{R}(H(x)) + B_{1m}L_f^m(b(h(x))) \quad \forall x \in \mathbf{cl}(\mathcal{O}).$$

Consequently, with (3.28) and (3.30), we finally obtain, for each x in $\mathbf{cl}(\mathcal{O})$,

$$\begin{aligned} L_f T(x) &= (kA)^{-m}L_f\mathfrak{R}(H(x)) - K^{-1}SL_f H(x), \\ &= kA[(kA)^{-m}\mathfrak{R}(H(x)) - K^{-1}SH(x)] + B_{1m}b(h(x)), \\ &= kAT(x) + B_{1m}b(h(x)). \end{aligned}$$

This proves that the function T defined by (3.28) is solution of (2.19).

Our second step in this proof is to show that, by picking k large enough, the function T given by (3.28) is uniformly injective. To simplify the following notations, to a function f , we associate the function Δf as follows :

$$\Delta f(x_1, x_2) = f(x_1) - f(x_2) .$$

So, for instance, for each pair (x_1, x_2) in \mathcal{O}^2 , we have :

$$T(x_1) - T(x_2) = (kA)^{-m} \Delta(\mathfrak{R} \circ H)(x_1, x_2) + K^{-1} S \Delta H(x_1, x_2) .$$

With (3.21), (3.26) and (3.24), we get, for each (Y_1, Y_2) in $\mathbb{R}^{m \times p} \times \mathbb{R}^{m \times p}$,

$$\begin{aligned} |\Delta \mathfrak{R}(Y_1, Y_2)| &\leq \int_{-\infty}^0 \left| \exp(-skA) B_{1m} \left[\check{F}(\mathfrak{Y}(Y_1, s))_m - \check{F}(\mathfrak{Y}(Y_2, s))_m \right] \right| ds \\ &\leq (1+L) \int_{-\infty}^0 |\exp(-skA)| |B_{1m}| |\mathfrak{Y}(Y_1, s) - \mathfrak{Y}(Y_2, s)| ds \\ &\leq (1+L) |B_{1m}| |Y_1 - Y_2| \int_{-\infty}^0 \exp\left(-s \left[1 + L + k \max_i \{\operatorname{Re}(\lambda_i)\}\right]\right) ds \\ &\leq \frac{(1+L)|B_{1m}|}{-[1 + L + k \max_i \{\operatorname{Re}(\lambda_i)\}]} |Y_1 - Y_2| . \end{aligned}$$

This yields, for each pair (x_1, x_2) in $\operatorname{cl}(\mathcal{O})^2$:

$$\begin{aligned} |T(x_1) - T(x_2)| &\geq |K^{-1} S \Delta H(x_1, x_2)| - |(kA)^{-m} \Delta(\mathfrak{R} \circ H)(x_1, x_2)| , \\ &\geq \frac{|\Delta H(x_1, x_2)|}{|K| |S^{-1}|} - \frac{|(kA)^{-m}| (1+L) |B_{1m}|}{-[1 + L + k \max_i \{\operatorname{Re}(\lambda_i)\}]} |\Delta H(x_1, x_2)| , \\ &\geq k^{-m} \left(\frac{1}{|S^{-1}|} - \frac{|A|^{-m} (1+L) |B_{1m}|}{-[1 + L + k \max_i \{\operatorname{Re}(\lambda_i)\}]} \right) |H(x_1) - H(x_2)| . \end{aligned}$$

So, with (2.18), the function T is uniformly injectivity on $\operatorname{cl}(\mathcal{O})$ for all k large enough.

3.4. Proof of Theorem 2.10. Following the same arguments as in the proof of Theorem 2.2, continuity and uniform injectivity (2.20) of the function T_a on $\operatorname{cl}(\mathcal{O})$ imply that $T_a(\operatorname{cl}(\mathcal{O}))$ is closed and we can construct a continuous function $T_a^* : \mathbb{C}^{m \times p} \rightarrow \operatorname{cl}(\mathcal{O})$ satisfying :

$$(3.31) \quad |T_a^*(z) - x| \leq \rho(4|z - T_a(x)|) \quad \forall (x, z) \in \operatorname{cl}(\mathcal{O}) \times \mathbb{C}^{m \times p} .$$

This implies :

$$(3.32) \quad T_a^*(T_a(x)) = x \quad \forall x \in \operatorname{cl}(\mathcal{O}) .$$

Now, let us assume for the time being there exists a function $\mathfrak{F} : \mathbb{C}^{m \times p} \rightarrow \mathbb{C}^{m \times p}$ to be used in (2.25) and satisfying :

$$(3.33) \quad |\mathfrak{E}(x) - \mathfrak{F}(z)| \leq N |T_a(x) - z| \quad \forall (x, z) \in \operatorname{cl}(\mathcal{O}) \times \mathbb{C}^{m \times p} .$$

As a direct consequence, we get the inequality :

$$|\mathfrak{F}(z)| \leq N |z| + M \quad z \in \mathbb{C}^{m \times p} ,$$

for some real number $M (= |\mathfrak{E}(x_0)| + N|T_a(x_0)|$, with some arbitrarily fixed x_0 in $\text{cl}(\mathcal{O})$). It follows that the z dynamics in the system (2.25) satisfy :

$$|\dot{z}| \leq (|A| + N)|z| + (M + |B(h(x))|).$$

Hence, for each x in \mathcal{O} and z in $\mathbb{C}^{m \times p}$, the component $Z(x, z, t)$ of a solution $(X(x, t), Z(x, z, t))$ of (2.25) is defined as long as $h(X(x, t))$ is defined. So this solution is right maximally defined on the same interval $[0, \sigma_{\mathbb{R}^n}^+(x))$ as $X(x, t)$ solution of (1.1).

With (3.31), (2.26) holds if we have :

$$(3.34) \quad \lim_{t \rightarrow \sigma_{\mathbb{R}^n}^+(x)} T_a(X(x, t)) - Z(x, z, t) = 0 \quad \forall (x, z) \in \mathcal{O} \times \mathbb{C}^{m \times p}.$$

To establish this limit, we associate, to each pair (x, z) in $\mathcal{O} \times \mathbb{C}^{m \times p}$, the matrix e in $\mathbb{C}^{m \times p}$:

$$e = T_a(x) - z.$$

With (2.21) and (2.25), we get :

$$\dot{e} = Ae + \mathfrak{E}(x) - \mathfrak{F}(z).$$

Let $U : \mathbb{C}^{m \times p} \rightarrow \mathbb{R}_+$ be the positive definite and proper function defined as :

$$U(e) = \sum_{i=1}^p \bar{e}_i^\top P e_i,$$

where e_i denotes the i^{th} column of e , \bar{e}_i denotes its complex conjugate and P is given by (2.24). Using (3.33) and completing the squares, we get :

$$\begin{aligned} \dot{\overline{U(e)}} &= \sum_{i=1}^p [-|e_i|^2 + 2\bar{e}_i^\top P (\mathfrak{E}(x) - \mathfrak{F}(z))_i] \leq -[1 - 2N\lambda_{\max}(P)]|e|^2, \\ &\leq -\frac{1 - 2N\lambda_{\max}(P)}{\lambda_{\min}(P)} U(e). \end{aligned}$$

So, with (2.23), we have established the existence of a strictly positive real number ε such that we have :

$$(3.35) \quad \dot{\overline{U(e)}} \leq -\varepsilon U(e) \quad \forall (x, z) \in \mathcal{O} \times \mathbb{C}^{m \times p}.$$

This implies, for all t in $[0, \sigma_{\mathcal{O}}^+(x))$ and (x, z) in $\mathcal{O} \times \mathbb{C}^{m \times p}$,

$$(3.36) \quad \exp(-\varepsilon t) U(e) \geq U(E(x, z, t)) \quad (= U(T_a(X(x, t)) - Z(x, z, t))).$$

With forward completeness within \mathcal{O} and the condition in the left of (2.26) (see (2.8)), this implies (3.34) holds.

It remains to establish the existence of a function $\mathfrak{F} : \mathbb{C}^{m \times p} \rightarrow \mathbb{C}^{m \times p}$ satisfying (3.33). With (3.32), we see that (2.22) becomes :

$$|\mathfrak{E}(T_a^*(z_1)) - \mathfrak{E}(T_a^*(z_2))| \leq N|z_1 - z_2| \quad \forall (z_1, z_2) \in T_a(\text{cl}(\mathcal{O}))^2.$$

Thus, $\mathfrak{E} \circ T_a^*$ is a Lipschitz function on the closed subset $T_a(\mathbf{cl}(\mathcal{O}))$. From Kirszbraun's Lipschitz extension Theorem, $\mathfrak{E} \circ T_a^*$ can be extended as a function $\mathfrak{F} : \mathbb{C}^{m \times p} \rightarrow \mathbb{C}^{m \times p}$ satisfying :

$$\begin{aligned} |\mathfrak{F}(z_1) - \mathfrak{F}(z_2)| &\leq N |z_1 - z_2| \quad \forall (z_1, z_2) \in (\mathbb{C}^{m \times p})^2, \\ \mathfrak{F}(z) &= \mathfrak{E}(T^*(z)) \quad \forall z \in T_a(\mathbf{cl}(\mathcal{O})) . \end{aligned}$$

So, in particular, we get (3.33).

3.5. Proof of Corollary 2.12. Let λ_i be the eigen values of a given diagonal Hurwitz complex $m \times m$ matrix A . With the notations (3.29), the function $T_a : \mathbf{cl}(\mathcal{O}) \rightarrow \mathbb{R}^{m \times p}$ defined in (2.27) can be rewritten as :

$$(3.37) \quad T_a(x) = -K^{-1}SH(x) .$$

In the following we show that we can find a real number $k^* \geq 1$ such that, if k is strictly larger than k^* , then the triple (kA, T_a, B_{1m}) satisfies all the assumptions of Theorem 2.10 :

1. The forward completeness within \mathcal{O} is satisfied by assumption.
2. (2.20) is satisfied since, using (2.18) and the definition of T_a in (3.37), we get, for each pair (x_1, x_2) in $\mathbf{cl}(\mathcal{O})^2$,

$$\begin{aligned} |x_1 - x_2| &\leq \rho(|S^{-1}K| |K^{-1}SH(x_1) - K^{-1}SH(x_2)|) , \\ &\leq \rho(|S^{-1}K| |T_a(x_1) - T_a(x_2)|) . \end{aligned}$$

3. Let the function $\mathfrak{E} : \mathbf{cl}(\mathcal{O}) \rightarrow \mathbb{R}^{m \times p}$ be defined as :

$$\mathfrak{E}(x) = -(kA)^{-m} B_{1m} L_f^m b(h(x)) .$$

We have to show that this function satisfies (2.21) and (2.22). Using (3.30), we get, for each x in \mathcal{O} ,

$$\mathfrak{E}(x) + B_{1m}b(h(x)) = -K^{-1}SL_fH(x) + kAK^{-1}SH(x) = L_fT_a(x) - kAT_a(x) .$$

So, (2.21) does hold. Also, with (2.17) and $k^{-m}|K| \leq 1$, which holds for $k \geq 1$, we get, for each (x_1, x_2) in $\mathbf{cl}(\mathcal{O})^2$,

$$\begin{aligned} |\mathfrak{E}(x_1) - \mathfrak{E}(x_2)| &= |(kA)^{-m} (B_{1m}(L_f^m(h(x_1)) - L_f^m(h(x_2))))| , \\ &\leq |(kA)^{-m}| |B_{1m}| L |H(x_1) - H(x_2)| , \\ &\leq \frac{1}{\min_i |\lambda_i|^m} |B_{1m}| L |S^{-1}| |T_a(x_1) - T_a(x_2)| . \end{aligned}$$

Hence, (2.22) is satisfied with $N = \frac{1}{\min_i |\lambda_i|^m} |B_{1m}| L |S^{-1}|$ which does not depend on k .

4. It remains to show that, by choosing k sufficiently large, the constraint (2.23) is satisfied. As kA is a diagonal complex matrix, the inequality (2.23) is simply :

$$\frac{1}{\min_i |\lambda_i|^m} |B_{1m}| L |S^{-1}| \frac{1}{k(-\max_i \operatorname{Re}(\lambda_i))} < 1 .$$

Clearly this inequality holds for all k large enough.

3.6. Technical comments on section 2.6. Due to space limitations, we give here only some hints on how the results established for the case of completeness can be extended to the case of boundedness observability.

The introduction of γ in the observer has mainly two consequences :

1. For the error convergence, t , in $\exp(At)$ in (3.1) or $\exp(-\varepsilon t)$ in (3.36), is replaced by the integral $\int_0^t \gamma(h(X(x, s)))ds$. If $\sigma_{\mathbb{R}^n}^+(x) = +\infty$, then, γ being larger than 1, this integral goes to $+\infty$ as t goes to $\sigma_{\mathbb{R}^n}^+(x)$. If, instead, $\sigma_{\mathbb{R}^n}^+(x)$ is finite, then $V_f(X(x, t))$ goes to $+\infty$ as t goes to $\sigma_{\mathbb{R}^n}^+(x)$. From (2.30) this is possible only if the above integral tends again to $+\infty$.

2. The function T given in (2.15) is defined in terms of the solutions $\check{X}(x, t)$ of the modified system (2.10) with f_γ instead of f . So we must show that this latter system shares the backward \mathcal{O} -distinguishability property of the original system (1.1). This can be done by associating, to each x in $\mathcal{O} + \delta_d$, the function $\tau_x : (\sigma_{\mathcal{O}+\delta_d}^-(x), \sigma_{\mathcal{O}+\delta_d}^+(x)) \rightarrow \mathbb{R}$ defined as :

$$\tau_x(t) = \int_0^t \gamma(h(X(x, s)))ds .$$

It admits an inverse τ^{-1} which is such that we have :

$$X(x, \tau_x^{-1}(t)) = \check{X}(x, t) \quad \forall x \in \mathcal{O} + \delta_d, \quad \forall t \in \tau_x(\sigma_{\mathcal{O}+\delta_d}^-(x), \sigma_{\mathcal{O}+\delta_d}^+(x)) .$$

Then it is possible to prove that, if, for some pair (x_1, x_2) in \mathcal{O}^2 , we have :

$$h(\check{X}(x_1, t)) = h(\check{X}(x_2, t)) \quad \forall t \in (\check{\sigma}_{\mathcal{O}+\delta_d}^-(x_1), 0] \cap (\check{\sigma}_{\mathcal{O}+\delta_d}^-(x_2), 0] ,$$

then we have also :

$$\tau_{x_1}^{-1}(t) = \tau_{x_2}^{-1}(t) \quad \forall t \in (\check{\sigma}_{\mathcal{O}+\delta_d}^-(x_1), 0] \cap (\check{\sigma}_{\mathcal{O}+\delta_d}^-(x_2), 0] .$$

4. Conclusion. We have stated sufficient conditions under which the extension to non linear systems of the Luenberger observer, as it has been proposed by Kazantzis and Kravaris in [13], can be used as long as the state to be observed remains in a given open set. In doing so, we have exploited the fact, already mentioned in [4, 16], that the observer proposed by Kreisselmeier and Engel in [14] is a possible way of implementing the Kazantzis-Kravaris / Luenberger observer.

We have established that a sufficient (row) dimension of the dynamic system giving the observer is $2 +$ twice the dimension of the state to be observed. This is in agreement with many other results known on the generic number of pieces of information to be extracted from the output paths to be able to reconstruct the state.

We have also shown that it is sufficient to know only an approximation of a solution of a partial differential equation which we need to solve to implement the observer. In this way, we have been able to make a connection with high gains observers.

finally, to get less restrictive sufficient conditions, we have found useful to modify the observer in a way which induces a time rescaling as already suggested in [4].

At this stage, our results are mainly of theoretical nature. They are concerned with existence. Several problems of prime importance for practice remain to be addressed like type and speed of convergence. In these regards, the contribution of Rapaport and Maloum in [18] is an important starting point.

Even for the purpose of showing the existence, we have to note that the conditions we have given can be strongly relaxed if an estimation of the norm of the state is available. This idea has been exploited in [4] where a truly global observer has been proposed under the assumption of global complete observability and unboundedness observability.

Appendix. Proof of Coron's Lemma 3.2. The idea of the proof is to show that the set

$$S = \bigcup_{x \in \Upsilon} \{ \Lambda = (\lambda_1, \dots, \lambda_{n+1}) \in \Omega^{n+1} : g(x, \lambda_\ell) = 0 \quad \forall \ell \in \{1, \dots, n+1\} \} ,$$

defined in (3.17) is contained in a countable union of sets which have zero Lebesgue measure.

Given $(\underline{x}, \underline{\Lambda}, \epsilon)$ in $\Upsilon \times \Omega^{n+1} \times \mathbb{R}_{+*}$, we denote by $S_{\epsilon, \underline{x}, \underline{\Lambda}}$ the set :

$$(A.1) \quad S_{\epsilon, \underline{x}, \underline{\Lambda}} = \bigcup_{x \in \mathcal{B}_\epsilon(\underline{x})} \{ \Lambda \in \mathcal{B}_\epsilon(\underline{\Lambda}) : g(x, \lambda_\ell) = 0 \quad \forall \ell \in \{1, \dots, n+1\} \} .$$

Assume for the time being that, for each pair $(\underline{x}, \underline{\Lambda})$ in $\Upsilon \times \Omega^{n+1}$, we can find a strictly positive real number ϵ and a countable family of C^1 functions $\sigma_i : \mathcal{B}_\epsilon(\underline{x}) \rightarrow \Omega^{n+1}$, such that we have :

$$(A.2) \quad S_{\epsilon, \underline{x}, \underline{\Lambda}} \subset \bigcup_{i \in \mathbb{N}} \sigma_i(\mathcal{B}_\epsilon(\underline{x})) .$$

The family $(\mathcal{B}_\epsilon(\underline{x}) \times \mathcal{B}_\epsilon(\underline{\Lambda}))_{(\underline{x}, \underline{\Lambda}) \in \Upsilon \times \Omega^{n+1}}$ is a covering of $\Upsilon \times \Omega^{n+1}$ by open subsets. From Lindelöf Theorem (see [6, Lemma 4.1] for instance), there exists a countable family $\{(\underline{x}_j, \underline{\Lambda}_j)\}_{j \in \mathbb{N}}$ such that we have :

$$\Upsilon \times \Omega^{n+1} \subset \bigcup_{j \in \mathbb{N}} \mathcal{B}_{\epsilon_j}(\underline{x}_j) \times \mathcal{B}_{\epsilon_j}(\underline{\Lambda}_j) ,$$

where ϵ_j denotes the ϵ associated to the pair $(\underline{x}_j, \underline{\Lambda}_j)$. With (A.2), it follows that we have :

$$S \subset \bigcup_{j \in \mathbb{N}} \bigcup_{i \in \mathbb{N}} \sigma_{i,j}(\mathcal{B}_{\epsilon_j}(\underline{x}_j)) ,$$

where $\sigma_{i,j}$ denotes the i th function σ associated with the pair $(\underline{x}_j, \underline{\Lambda}_j)$. The set $\sigma_{i,j}(\mathcal{B}_{\epsilon_j}(\underline{x}_j))$ is the image, contained in \mathbb{C}^{n+1} , a real manifold of dimension $2(n+1)$, by a C^1 function of $\mathcal{B}_{\epsilon_j}(\underline{x}_j)$, a real manifold of dimension $2n$. From a variation on Sard's Theorem (see [20, Theorem 3, paragraphe 3] for instance), this image $\sigma_{i,j}(\mathcal{B}_{\epsilon_j}(\underline{x}_j))$ has zero Lebesgue measure in \mathbb{C}^{n+1} . So S , being a countable union of such zero Lebesgue measure subsets, has zero Lebesgue measure.

So all we have to do to establish Lemma 3.2 is to prove the existence of ε and the functions σ_i satisfying (A.2) for each pair $(\underline{x}, \underline{\Lambda})$ in $\Upsilon \times \Omega^{n+1}$. For ε , we consider two cases :

1. Consider a pair $(\underline{x}, \underline{\Lambda})$ such that $g_j(\underline{x}, \underline{\lambda}_\ell)$ is non zero for some component λ_ℓ of $\underline{\Lambda}$ and g_j of g . By continuity of g_j , we can find a strictly positive real number ϵ such that $g(x, \lambda_\ell)$ is also non zero for all x in $\mathcal{B}_\epsilon(\underline{x})$ and Λ in $\mathcal{B}_\epsilon(\underline{\Lambda})$. In this case, the set $S_{\epsilon, \underline{x}, \underline{\Lambda}}$ is empty.

2. Consider a pair $(\underline{x}, \underline{\Delta})$ such that $g(\underline{x}, \underline{\Delta}_\ell)$ is zero for each of the $n+1$ components of $\underline{\Delta}_\ell$ of $\underline{\Delta}$. From the assumption (3.16), for each ℓ , we can find a component g_{j_ℓ} of g and an integer k_ℓ satisfying :

$$\frac{\partial^i g_{j_\ell}}{\partial \lambda^i}(\underline{x}, \underline{\Delta}_\ell) = 0 \quad \forall i \in \{0, \dots, k_\ell - 1\} \quad , \quad \frac{\partial^{k_\ell} g_{j_\ell}}{\partial \lambda^{k_\ell}}(\underline{x}, \underline{\Delta}_\ell) \neq 0 .$$

In this case, following the Weierstrass Preparation Theorem (see [12, Theorem IV.1.1]² for instance), for each ℓ in $\{1, \dots, n+1\}$, we know the existence of a strictly positive real number ϵ_ℓ , a function $q_\ell : \mathcal{B}_{\epsilon_\ell}(\underline{x}) \times \mathcal{B}_{\epsilon_\ell}(\underline{\Delta}_\ell) \rightarrow \mathbb{C}$, and k_ℓ C^1 functions $a_j^\ell : \mathbb{R}^{2n} \rightarrow \mathbb{C}$ satisfying, for all (x, λ) in $\mathcal{B}_{\epsilon_\ell}(\underline{x}) \times \mathcal{B}_{\epsilon_\ell}(\underline{\Delta}_\ell)$,

$$(A.3) \quad q_\ell(x, \lambda) g_{j_\ell}(x, \lambda) = (\lambda - \underline{\Delta}_\ell)^{k_\ell} + \sum_{j=0}^{k_\ell-1} a_j^\ell(x) (\lambda - \underline{\Delta}_\ell)^j .$$

We choose the real number ϵ , to be associated to $(\underline{x}, \underline{\Delta})$ in the definition of $S_{\epsilon, \underline{x}, \underline{\Delta}}$, as :

$$\epsilon = \inf_{\ell \in \{1, \dots, n+1\}} \epsilon_\ell .$$

In the following $P_\ell : \mathcal{B}_\epsilon(\underline{x}) \times \mathbb{C} \rightarrow \mathbb{C}$ and $a^\ell(x) : \mathcal{B}_\epsilon(\underline{x}) \rightarrow \mathbb{C}^{k_\ell}$ denote :

$$P_\ell(x, \lambda) = (\lambda - \underline{\Delta}_\ell)^{k_\ell} + \sum_{j=0}^{k_\ell-1} a_j^\ell(x) (\lambda - \underline{\Delta}_\ell)^j \quad , \quad a^\ell(x) = (a_0^\ell(x), \dots, a_{k_\ell-1}^\ell(x)) .$$

With this definition of ϵ , we have the following implication, for Λ in $\mathcal{B}_\epsilon(\underline{\Delta})$ and x in $\mathcal{B}_\epsilon(\underline{x})$,

$$(A.4) \quad g(x, \lambda_\ell) = 0 \quad \forall \ell \in \{1, \dots, n+1\} \quad \Rightarrow \quad (\lambda_\ell, a^\ell(x)) \in M^\ell \quad \forall \ell \in \{1, \dots, n+1\}$$

where M^ℓ is the set :

$$(A.5) \quad M^\ell = \left\{ (\lambda, (b_0, \dots, b_{k_\ell-1})) \in \mathbb{C} \times \mathbb{C}^{k_\ell} : (\lambda - \underline{\Delta}_\ell)^{k_\ell} + \sum_{j=0}^{k_\ell-1} b_j (\lambda - \underline{\Delta}_\ell)^j = 0 \right\}$$

Our interest in this set follows from the following Lemma, which we prove later on,

LEMMA A.1. *Let M be the set defined as :*

$$M = \left\{ (\lambda, b_0, \dots, b_{k-1}) \in \mathbb{C} \times \mathbb{C}^k : \lambda^k + \sum_{j=0}^{k-1} b_j \lambda^j = 0 \right\} .$$

There exists a countable family $\{M_m\}_{m \in \mathbb{N}}$ of regular submanifolds of \mathbb{C}^k and a countable family of C^1 functions $\rho_m : M_m \rightarrow \mathbb{C}$ such that we have the inclusion :

$$(A.6) \quad M \subset \bigcup_{m \in \mathbb{N}} \bigcup_{b \in M_m} \{(\rho_m(b), b)\} .$$

²In [12, Theorem IV.1.1], this theorem is stated with the assumption that g_j is holomorphic in both x and λ . However, as far as x is concerned, it can be seen in the proof of this Theorem that we need only the implicit function theorem to apply. So continuous differentiability in x for each λ is enough.

So, for each ℓ in $\{1, \dots, n+1\}$ we have a countable family $\{M_{m_\ell}^\ell\}_{m_\ell \in \mathbb{N}}$ of regular submanifolds of \mathbb{C}^{k_ℓ} and a countable family of C^1 functions $\rho_{m_\ell}^\ell : M_{m_\ell}^\ell \rightarrow \mathbb{C}$ such that, for each x in $\mathcal{B}_\epsilon(\underline{x})$, if $P_\ell(x, \lambda_\ell)$ is zero, then there exists an integer m_ℓ such that we have :

$$(A.7) \quad a^\ell(x) \in M_{m_\ell}^\ell, \quad \lambda_\ell = \rho_{m_\ell}^\ell(a^\ell(x)).$$

Hence, with (A.4), if :

$$g(x, \lambda_\ell) = 0 \quad \forall \ell \in \{1, \dots, n+1\},$$

then there exists an $(n+1)$ -tuple $\mu = (m_1, \dots, m_{n+1})$ of integers satisfying :

$$a^\ell(x) \in M_{m_\ell}^\ell, \quad \lambda_\ell = \rho_{m_\ell}^\ell(a^\ell(x)) \quad \forall \ell \in \{1, \dots, n+1\}.$$

So, by letting :

$$(A.8) \quad S_{\epsilon, \underline{x}, \underline{\Delta}}^\mu = \bigcup_{\{x \in \mathcal{B}_\epsilon(\underline{x}) : a^\ell(x) \in M_{m_\ell}^\ell \quad \forall \ell \in \{1, \dots, n+1\}\}} \left\{ (\rho_{m_1}^1(a^1(x)), \dots, \rho_{m_{n+1}}^{n+1}(a^{n+1}(x))) \right\}$$

we have established :

$$(A.9) \quad S_{\epsilon, \underline{x}, \underline{\Delta}} \subset \bigcup_{\mu \in \mathbb{N}^{n+1}} S_{\epsilon, \underline{x}, \underline{\Delta}}^\mu.$$

Comparing (A.2) with (A.9) and using the definition (A.8), we see that a candidate for the function σ_i is :

$$\sigma_i(x) = \left(\rho_{m_\ell}^\ell \left(R_{M_{m_\ell}^\ell} (a^\ell(x)) \right) \right)_{\ell \in \{1, \dots, n+1\}}$$

where i happens to be the $(n+1)$ -tuple μ and $R_{M_{m_\ell}^\ell} : \mathbb{C}^{k_\ell} \rightarrow M_{m_\ell}^\ell$ is a “restriction” to $M_{m_\ell}^\ell$ since we have to consider only those $a^\ell(x)$ which are in $M_{m_\ell}^\ell$. Finding such functions $R_{M_{m_\ell}^\ell}$ such that σ_i is C^1 may not be possible. But, following [7, Lemma 3.3], we know the existence, for each ℓ , of a countable family of C^1 functions $R_\nu^\ell : \mathbb{C}^{k_\ell} \rightarrow M_{m_\ell}^\ell$ such that we get :

$$S_{\epsilon, \underline{x}, \underline{\Delta}}^\mu \subset \bigcup_{\nu \in \mathbb{N}} \left\{ \left(\rho_{m_\ell}^\ell \left(R_\nu^\ell (a^\ell(\mathcal{B}_\epsilon(\underline{x}))) \right) \right)_{\ell \in \{1, \dots, n+1\}} \right\}.$$

In other words the family of functions σ_i is actually given by the family :

$$\sigma_{\mu, \nu} = \left(\rho_{m_\ell}^\ell \circ R_\nu^\ell \circ a_\ell \right)_{\ell \in \{1, \dots, n+1\}}$$

i.e. we have :

$$S_{\epsilon, \underline{x}, \underline{\Delta}} \subset \bigcup_{\mu \in \mathbb{N}^{n+1}} \bigcup_{\nu \in \mathbb{N}} \sigma_{\mu, \nu}(\mathcal{B}_\epsilon(\underline{x})). \quad \square$$

Proof of Lemma A.1 : The inclusion (A.6) says that we are looking for a covering of the set M with some special structure. A covering easily found, but not having

this special structure, is obtained by choosing a first complex number λ , denoted λ_1 , as well as $k - 1$ other complex numbers λ_j . Then the b_j 's are given by the identity :

$$(A.10) \quad \prod_{j=1}^k (\lambda - \lambda_j) = \lambda^k + \sum_{j=0}^{k-1} b_j \lambda^j, \quad \lambda \in \mathbb{C}.$$

In other words if we denote by $\eta : \mathbb{C}^k \rightarrow \mathbb{C}^k$ the function which gives the b_j 's from the λ_j 's, we have :

$$\bigcup_{(\lambda_1, \dots, \lambda_k) \in \mathbb{C}^k} \{(\lambda_1, \eta(\lambda_1, \dots, \lambda_k))\} \subseteq M.$$

Specifically, given the elementary symmetric functions s_i , sum of all the products of i distinct λ_j 's,

$$s_i = \lambda_1 \dots \lambda_{i-1} \lambda_i + \lambda_1 \dots \lambda_{i-1} \lambda_{i+1} + \dots + \lambda_{k-i+1} \lambda_{k-i+2} \dots \lambda_k,$$

the b_j 's are obtained as :

$$b_j = (-1)^{k-j} s_{k-j} \quad \forall j \in \{1, \dots, k-1\}.$$

Also the elementary symmetric functions are related to the sum of similar powers σ_p :

$$\sigma_p = \sum_{j=1}^k \lambda_j^p,$$

via the Newton equations :

$$\sigma_i - \sigma_{i-1} s_1 + \sigma_{i-2} s_2 - \dots + (-1)^{i-1} \sigma_1 s_{k-1} + (-1)^i i s_i = 0.$$

The corresponding functions $(\sigma_p) \mapsto (s_i)$ and $(s_i) \mapsto (b_j)$ are C^∞ diffeomorphisms.

To obtain the result stated in the Lemma, we need to invert the function $\eta : (\lambda_\ell) \mapsto (b_j)$. This function is known to be a homeomorphism if the λ_j 's are defined up to permutations (See [5, Proposition 1.5.5] for instance). But unfortunately we cannot go beyond continuity of the inverse because of possible multiple roots. To round this problem, we choose the multiplicity c_ℓ of the root λ_ℓ so that the sum of the c_ℓ 's is k . So, except if they are all 1, some of them must be 0. Maybe after re-ordering, we can assume that each c_1 to c_q is non zero and satisfy :

$$c_1 + \dots + c_q = k.$$

Then we choose q different complex numbers ϖ_ℓ and we let :

$$\begin{aligned} \lambda_1 = \dots = \lambda_{c_1} = \varpi_1, \quad \lambda_{c_0+1} = \dots = \lambda_{c_1+c_2} = \varpi_2, \quad \dots \\ \dots, \quad \lambda_{c_1+\dots+c_{q-1}+1} = \dots = \lambda_{c_1+\dots+c_q} = \varpi_q. \end{aligned}$$

This yields :

$$(A.11) \quad \sigma_p = \sum_{\ell=1}^q c_\ell \varpi_\ell^p.$$

We stress at this point that, to any k -tuple of λ_ℓ 's in \mathbb{C}^k , we can associate, maybe after a permutation θ of its components, such q -tuples of $c = (c_r)$ and $\varpi = (\varpi_r)$, with $\varpi_i \neq \varpi_j$ if $i \neq j$. It follows that the function η can be decomposed as follows :

$$(\lambda_\ell) \mapsto \underbrace{\theta(\lambda_\ell) \mapsto (c, \varpi) \mapsto (\sigma_p) \mapsto (s_i) \mapsto (b_j)}_{\eta} .$$

This way, given a permutation θ and a root multiplicity vector c , with no zero component, we have defined a function $\gamma : \mathbb{C}^q \setminus \{\varpi_i = \varpi_j\} \rightarrow \mathbb{C}^k$ which maps the ϖ_r 's into the b_j 's :

$$\gamma : \varpi \mapsto (\sigma_p) \mapsto (s_i) \mapsto (b_j) .$$

This function has rank q . Indeed we know that the last two functions above are diffeomorphisms and, for the first one, we get from (A.11) :

$$\frac{\partial \sigma_p}{\partial \varpi_r} = p c_r \varpi_r^{p-1} .$$

Since the p 's and c_r 's are not zero, we see that the matrix $\left(\frac{\partial \sigma_p}{\partial \varpi_r}\right)$ has full rank since it has a Vandermonde structure and the ϖ_r 's are different. Consequently, the jacobian matrix $\left(\frac{\partial b_j}{\partial \varpi_r}\right)$ of the function γ has full rank $q \leq k$. It follows from [6, Theorem III,4.12, Theorem III, 5.5] that, for each q -tuple ϖ in $\mathbb{C}^q \setminus \{\varpi_i = \varpi_j\}$, there exists a strictly positive real number $\epsilon(\varpi)$ such that

- $\mathcal{B}_{\epsilon(\varpi)}(\varpi)$ is a subset of $\mathbb{C}^q \setminus \{\varpi_i = \varpi_j\}$,
- $\gamma(\mathcal{B}_{\epsilon(\varpi)}(\varpi))$ is a regular submanifold of (the real manifold) \mathbb{C}^k ,
- the restriction of $\gamma : \mathcal{B}_{\epsilon(\varpi)}(\varpi) \rightarrow \gamma(\mathcal{B}_{\epsilon(\varpi)}(\varpi))$ is a diffeomorphism.

We denote by γ^{-1} the “inverse” function.

The family $\{\mathcal{B}_{\epsilon(\varpi)}(\varpi)\}_{\varpi \in \mathbb{C}^q \setminus \{\varpi_i = \varpi_j\}}$ is a covering by open subsets of $\mathbb{C}^q \setminus \{\varpi_i = \varpi_j\}$. So there exists a countable family $(\varpi^i)_{i \in \mathbb{N}}$ such that the family $\{\mathcal{B}_{\epsilon(\varpi^i)}(\varpi^i)\}_{i \in \mathbb{N}}$ is covering by open subsets of $\mathbb{C}^q \setminus \{\varpi_i = \varpi_j\}$. Moreover, since, to each k -tuple (b_j) in \mathbb{C}^k , we can associate a pair (c, ϖ) with $\varpi_i \neq \varpi_j$ if $i \neq j$, any such k -tuple (b_j) is in at least one set $\gamma(\mathcal{B}_{\epsilon(\varpi^i)}(\varpi^i))$. So, since the number of permutations θ in \mathbb{C}^k and multiplicity vectors c is finite, with varying i and q , we get a countable family $\{M_m\}_{m \in \mathbb{N}}$ of regular submanifold of \mathbb{C}^k defined as :

$$M_m := \gamma(\mathcal{B}_{\epsilon(\varpi^i)}(\varpi^i))$$

and a countable family of C^1 functions ρ_m defined as :

$$\rho_m : (b_0, \dots, b_{k-1}) \in M_m \xrightarrow{\gamma^{-1}} \varpi \in \mathcal{B}_{\epsilon(\varpi^i)}(\varpi^i) \xrightarrow{(c, \theta)} (\lambda_1, \dots, \lambda_k) \in \mathbb{C}^k .$$

Each $b = (b_0, \dots, b_{k-1})$ in \mathbb{C}^k is in least one M_m and we have :

$$\eta(\rho_m(b)) = b ,$$

Our result follows then from :

$$\mathbb{C}^k = \bigcup_{m \in \mathbb{N}} \bigcup_{b \in M_m} \{\rho_m(b)\} .$$

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